Closing Tue: 12.1, 12.2, 12.3
Closing Thu: 12.4(1), 12.4(2), 12.5(1)
Please carefully read my 12.3, 12.4 review sheets. Then look at the 12.5 visuals before class Wednesday.

12.4 The Cross Product

We define the cross product, or vector product, for two 3-dimensional vectors,
$\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and
$\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$,
by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$(a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Ex: $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(2 \cdot 2 - 0 \cdot 3)\boldsymbol{i} - (1 \cdot 2 - 0 \cdot (-1))\boldsymbol{j} + (1 \cdot 3 - 2 \cdot (-1))\boldsymbol{k}$$

$$\boxed{\langle 4, -2, 5 \rangle.}$$

NOTE:

$\langle 4, -2, 5 \rangle \cdot \langle 1, 2, 0 \rangle = 4 - 4 + 0$
$= 0$ ☆

$\langle 4, -2, 5 \rangle \cdot \langle -1, 3, 2 \rangle = -4 - 6 + 10$
$= 0$ ☆

You do: $\boldsymbol{a} = \langle 1,3,-1\rangle$, $\boldsymbol{b} = \langle 2,1,5\rangle$.

Compute $\boldsymbol{a} \times \boldsymbol{b}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{vmatrix}$$

$$= (15 - -1)\vec{i} - (5 - -2)\vec{j} + (1 - 6)\vec{k}$$

$$= \langle 16, -7, -5\rangle$$

NOTE: $\langle 16,-7,-5\rangle \cdot \langle 1,3,-1\rangle = 16 - 21 + 5 = 0$ ✓

$\langle 16,-7,-5\rangle \cdot \langle 2,1,5\rangle = 32 - 7 - 25 = 0$ ✓

Most important fact:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to *both* **a** and **b**.

$\boxed{\text{proof}}$

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 \; ?\,? \rangle \cdot \langle a_1, a_2, a_3 \rangle = 0$$

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ -a_2b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle = 0$$

$$a_1a_2b_3 - \cancel{a_1a_3b_2} + a_2a_3b_1 - a_1a_2b_3 + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1} = 0$$

$$a_2b_1b_3 - \cancel{a_3b_1b_2} + \cancel{a_3b_1b_2} - \cancel{a_1b_2b_3} + \cancel{a_1b_2b_3} - a_2b_1b_3 = 0$$

So you roughly see how someone could arrive at such a formula.

Note: If **a** and **b** are parallel to each other, then there are many vectors perpendicular to both **a** and **b**.
So what happens to $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$?

Example: Give me any two vectors that are parallel and let's see.

$\vec{a} = \langle 1, -3, 4 \rangle$ $\cdot 2$

$\vec{b} = \langle 2, -6, 8 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 4 \\ 2 & -6 & 8 \end{vmatrix}$$

$= (-24 - -24)\vec{i} - (8 - 8)\vec{j} + (-6 - -6)\vec{k}$

$= \langle 0, 0, 0 \rangle$

$$\boxed{\begin{array}{l} \vec{a} \times \vec{b} = \langle 0,0,0 \rangle = \vec{0} \\ \iff \vec{a} \text{ AND } \vec{b} \text{ ARE PARALLEL} \end{array}}$$

$$\boxed{\begin{array}{l} \vec{a} \cdot \vec{b} = 0 \\ \iff \vec{a} \text{ AND } \vec{b} \text{ ARE ORTHOGONAL} \end{array}}$$

Right-hand rule

If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of **a** $\times$ **b**.

$\boxed{\text{ORDER MATTERS}}$

$\langle 1, 2, 0\rangle \times \langle -1, 3, 2\rangle = \langle 4, -2, 5\rangle$

$\langle -1, 3, 2\rangle \times \langle 1, 2, 0\rangle = \langle -4, 2, -5\rangle$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

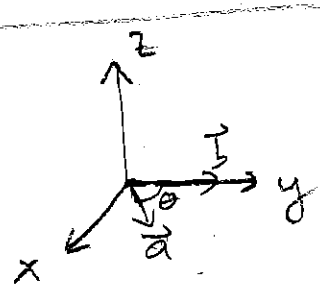
opposite direction!

Example 1

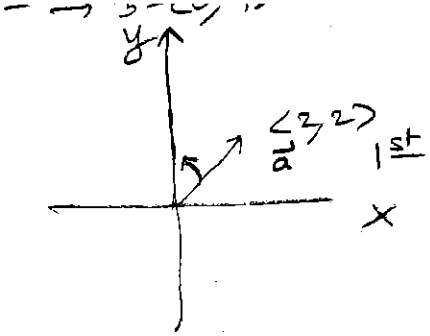
$\vec{a} = \langle 2, 2, 0\rangle$

$\vec{b} = \langle 0, 4, 0\rangle$

WILL $\vec{a} \times \vec{b}$ point upward or downward?



$\vec{a} \times \vec{b}$ WILL BE UPWARD!



CHECK:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 0 & 4 & 0 \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (8-0)\vec{k} = \langle 0, 0, 8\rangle$$

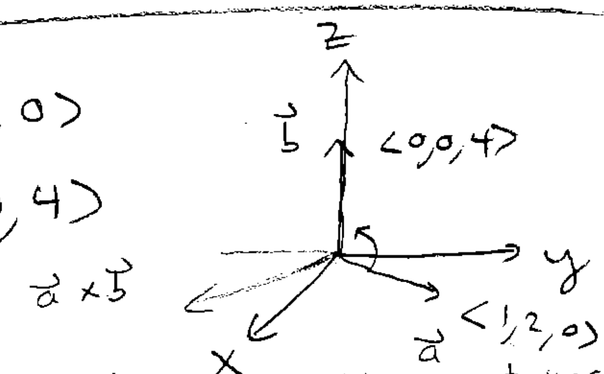
$+ \Rightarrow \boxed{\text{UPWARD}}$

$\vec{b} \times \vec{a}$ WOULD BE DOWNWARD $\langle 0, 0, -8\rangle$

EXAMPLE

$\vec{a} = \langle 1, 2, 0\rangle$

$\vec{b} = \langle 0, 0, 4\rangle$



$$\vec{a} \times \vec{b} = \langle \underbrace{\overset{\text{toward us}}{+}}_{+ \text{ or } -}, \underbrace{\overset{\text{toward left}}{-}}_{+ \text{ or } -}, \underbrace{\overset{\text{not up or down}}{0}}_{+ \text{ or } -} \rangle$$

CHECK

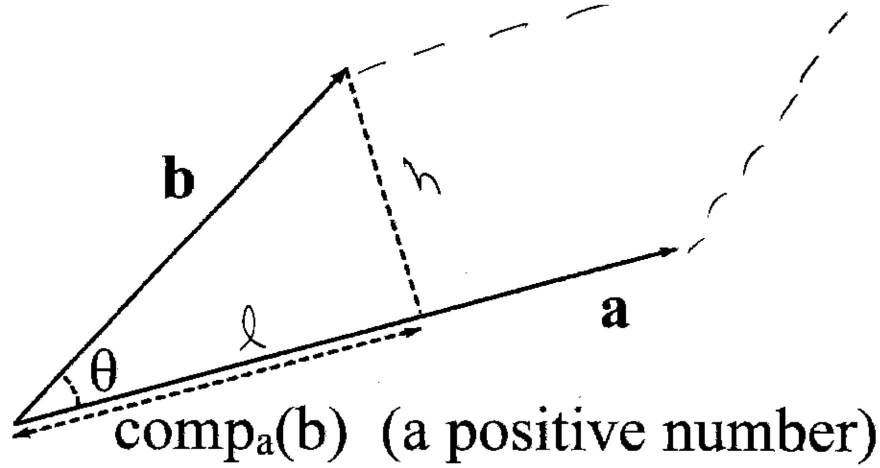
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix} = \langle 8, -8, 0\rangle \ \checkmark$$

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$:

Through some algebra and using the dot product rules, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$



$$\cos\theta = \frac{\ell}{|\vec{b}|}$$

$$\ell = |\vec{b}|\cos\theta$$

$$= |\vec{b}| \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\ell = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$\underbrace{\mathrm{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

REVIEW

(LAST TIME

$$\sin\theta = \frac{h}{|\vec{b}|}$$

$$h = |\vec{b}|\sin\theta$$

NOTE:

$\theta = 90°$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|$

RECTANGLE!

$\theta = 0$ or $\theta = 180°$

$\Rightarrow |\vec{a}\times\vec{b}| = 0$

$\vec{a}\times\vec{b} = \langle 0, 0, 0\rangle$

Note: $|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$ is the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$

AREA OF TRIANGLE FORMED BY $\vec{a}$ AND $\vec{b}$

$= \frac{1}{2}|\vec{a}\times\vec{b}|$.

Ex)

FIND THE AREA OF THE TRIANGLE FORMED BY

$P(3,0,3)$, $Q(-2,1,4)$, $R(7,2,5)$

$\langle 0,14,-14\rangle \cdot \langle -5,1,1\rangle = 0$ ✓

$\langle 0,14,-14\rangle \cdot \langle 4,2,2\rangle = 0$ ✓

CHECK!

$\vec{PQ} = \langle -5, 1, 1\rangle$

$\vec{PR} = \langle 4, 2, 2\rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & 1 \\ 4 & 2 & 2 \end{vmatrix} = (2-2)\vec{i} - (-10-4)\vec{j} + (-10-4)\vec{k}$$

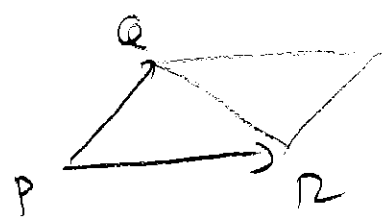
$$= \langle 0, 14, -14\rangle$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{0^2 + 14^2 + (-14)^2} = \sqrt{2 \cdot 14^2} = \boxed{14\sqrt{2}} \approx 19.79899$$

AREA OF PARALLELOGRAM

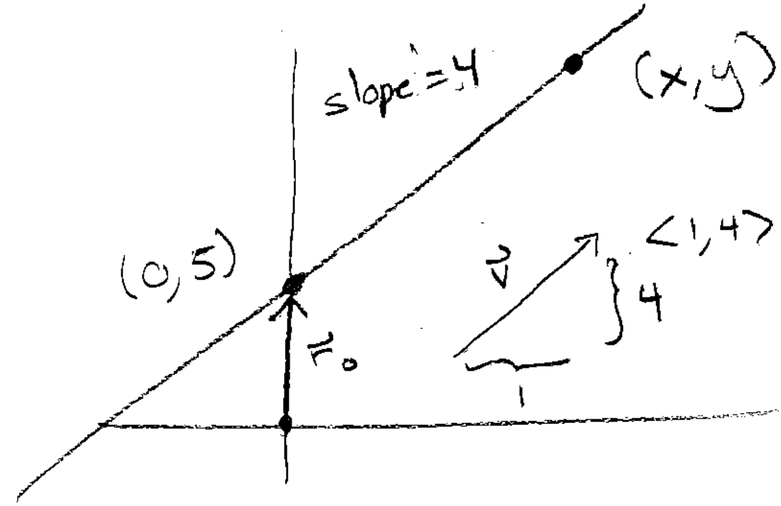
AREA OF TRIANGLE $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2}(14\sqrt{2}) = \boxed{7\sqrt{2}}$



12.5 Intro to Lines in 3D

To describe 3D lines we use parametric equations.



<u>Here is a 2D example</u>

Consider the 2D line: y = 4x + 5.

(a) Find a vector parallel to the line. Call it vector **v.**

$\vec{v} = \langle 1, 4 \rangle$ works

(b) Find a vector whose head touches some point on the line when drawn from the origin. Call it vector **r₀.**

$\vec{r}_0 = \langle 0, 5 \rangle$ works

(c) We can reach all other points on the line by walking along **r₀**, then adding scale multiples of **v**.

some multiple

$$\underbrace{\langle x, y \rangle}_{\vec{r}} = \langle 0, 5 \rangle + t \langle 1, 4 \rangle$$

$$\vec{r} = \vec{r}_0 + t\,\vec{v}$$

$$\boxed{\begin{aligned} x &= 0 + t \\ y &= 5 + 4t \end{aligned}}$$

This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:

$\boldsymbol{v} = \langle a, b, c \rangle$ = parallel to the line.

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = position vector

then all other points, (x, y, z), satisfy

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$,

for some number t.

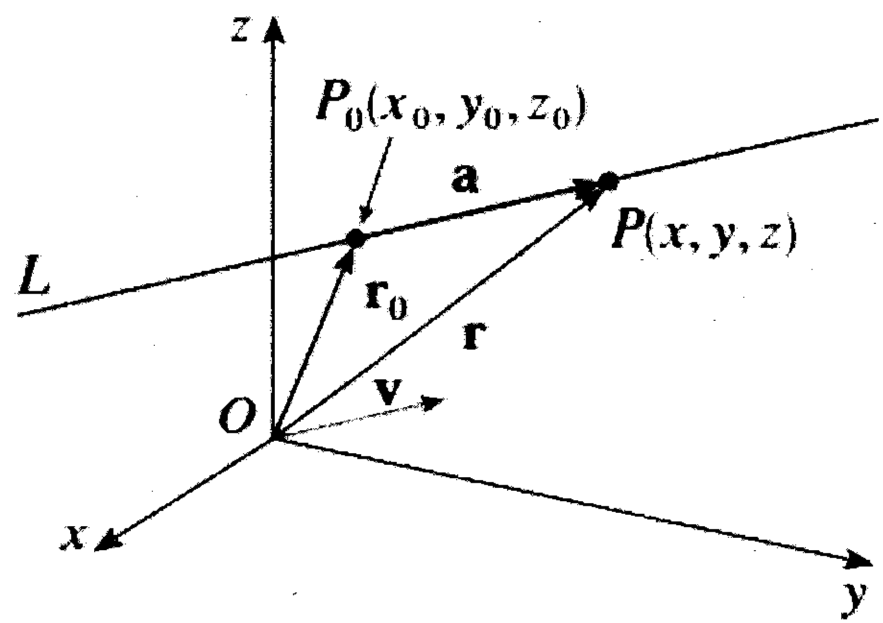
The above form ($\boldsymbol{r} = \boldsymbol{r_0} + t\,\boldsymbol{v}$) is called the *vector form* of the line.

We also can write this in *parametric form* as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

or in *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$



Basic Example – Given Two Points:
Find parametric equations of the line thru P(3, 0, 2) and Q(-1, 2, 7).

NEED

1. POSITION VECTOR : $\vec{r}_0 = \langle 3, 0, 2 \rangle$ (COULD ALSO HAVE USED $\langle -1, 2, 7 \rangle$)

2. DIRECTION VECTOR : $\vec{v} = \overrightarrow{PQ} = \langle -4, 2, 5 \rangle$ (COULD ALSO HAVE USED $\overrightarrow{QP}$)

$$\langle x, y, z \rangle = \langle 3, 0, 2 \rangle + t \langle -4, 2, 5 \rangle$$

$$\Rightarrow \boxed{\begin{aligned} x &= 3 - 4t \\ y &= 0 + 2t \\ z &= 2 + 5t \end{aligned}}$$

EVERY POINT (x, y, z) ON THE LINE SATISFIES THIS EQUATION FOR SOME VALUE OF t

$$t = \frac{x-3}{-4} \quad \text{AND} \quad t = \frac{y-0}{2} \quad \text{AND} \quad t = \frac{z-2}{5}$$

THUS,

$$\frac{x-3}{-4} = \frac{y-0}{2} = \frac{z-2}{5}$$

SIMULTANEOUSLY, IF (x, y, z) IS ON THE LINE

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

$\boxed{L1}$ $x = 3+2t$, $y = -7t$, $z = 10+t$

$\boxed{L2}$ $x = 14+6t$, $y = 3-21t$, $z = 18+3t$

PARALLEL

$\langle 2,-7,1\rangle$, $\langle 6,-21,3\rangle$ PARALLEL

2. Two lines **intersect** if they have an (x, y, z) point in common (use different parameters when you combine!)
Note: The *acute angle of intersection* is the acute angle between the direction vectors.

$\boxed{L1}$ $x = t$, $y = 1+2t$, $z = 2+3t$

$\boxed{L2}$ $x = 3-4u$, $y = 2-3u$, $z = 1+2u$

① $t = x \stackrel{?}{=} 3-4u \iff t = 3-4u$

② $1+2t = y \stackrel{?}{=} 2-3u \Rightarrow 1+2(3-4u) \stackrel{?}{=} 2-3u$

$7-8u = 2-3u$

$5 = 5u$

$u=1$

$t=-1$

③ $2+3t = z \stackrel{?}{=} 1+2u$

$-1 \neq 3$

NO, DO NOT INTERSECT

3. Two lines are **skew** if they don't intersect and aren't parallel.